

# HARMONICITY AND MINIMALITY OF VECTOR FIELDS ON FOUR-DIMENSIONAL LORENTZIAN LIE GROUPS

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**ABSTRACT.** We consider four-dimensional lie groups equipped with left-invariant Lorentzian Einstein metrics, and determine the harmonicity properties of vector fields on these spaces. In some cases, all these vector fields are critical points for the energy functional restricted to vector fields. We also classify vector fields defining harmonic maps, and calculate explicitly the energy of these vector fields. Then we study the minimality of critical points for the energy functional.

## 1. INTRODUCTION

In [3] it has been proved that a (simply connected) four-dimensional homogeneous Riemannian manifold is either symmetric, or isometric to a Lie group equipped with a left-invariant Riemannian metric. Following [2], in a sense, by Proposition 2.2 in [6] the classification of four-dimensional Lorentzian Lie groups coincide with the Riemannian ones which on that base, four-dimensional Einstein Lorentzian lie groups have been classified [6]. On the other hand, investigating critical points of the energy associated to vector fields is an interesting purpose under different points of view. As an example by the Reeb vector field  $\xi$  of a contact metric manifold, somebody can see how the criticality of such a vector field is related to the geometry of the manifold ([16],[17]). Recently, it has been [11] proved that critical points of  $E : \mathfrak{X}(M) \rightarrow R$ , that is, the energy functional restricted to vector fields, are again parallel vector fields. Moreover, in the same paper it also has been determined the tension field associated to a unit vector field  $V$ , and investigated the problem of determining when  $V$  defines a harmonic map. So it makes sense to determine the harmonicity properties of vector fields on four-dimensional Lorentzian Einstein lie groups.

A Riemannian manifold admitting a parallel vector field is locally reducible, and the same is true for a pseudo-Riemannian manifold admitting an either space-like or time-like parallel vector field. This leads us to consider different situations, where some interesting types of non-parallel vector fields can be characterized in terms of harmonicity properties (see [9], [14] and [15]) in this paper.

Let  $(M, g)$  be a compact pseudo-Riemannian manifold and  $g_s$  be the Sasaki metric on the tangent bundle  $TM$ , then the energy of a smooth vector field  $V : (M, g) \rightarrow (TM, g^s)$  on

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2000 *Mathematics Subject Classification.* 53C50, 53C15, 53C25.

*Key words and phrases.* Harmonic vector fields, Harmonic maps, Einstein metrics, Lie group, Pseudo-Riemannian homogeneous spaces.

$M$  is;

$$(1.1) \quad E(V) = \frac{n}{2} \text{vol}(M, g) + \frac{1}{2} \int_M \|\nabla V\|^2 dv$$

(assuming  $M$  compact; in the non-compact case, one works over relatively compact domains see [4]). If  $V : (M, g) \rightarrow (TM, g^s)$  be a critical point for the energy functional, then  $V$  is said to define a harmonic map. The Euler-Lagrange equations characterize vector fields  $V$  defining harmonic maps as the ones whose tension field  $\theta(V) = \text{tr}(\nabla^2 V)$  vanishes. Consequently,  $V$  defines a harmonic map from  $(M, g)$  to  $(TM, g^s)$  if and only if

$$(1.2) \quad \text{tr}[R(\nabla V, V)] = 0, \quad \nabla^* \nabla V = 0,$$

where with respect to a pseudo-orthonormal local frame  $\{e_1, \dots, e_n\}$  on  $(M, g)$ , with  $\varepsilon_i = g(e_i, e_i) = \pm 1$  for all indices  $i$ , one has

$$\nabla^* \nabla V = \sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V).$$

A smooth vector field  $V$  is said to be a harmonic section if and only if it is a critical point of  $E^v(V) = (1/2) \int_M \|\nabla V\|^2 dv$  where  $E^v$  is the vertical energy. The corresponding Euler-Lagrange equations are given by

$$(1.3) \quad \nabla^* \nabla V = 0,$$

Let  $\mathfrak{X}^\rho(M) = \{V \in \mathfrak{X}(M) : \|V\|^2 = \rho^2\}$  and  $\rho \neq 0$ . Then, one can consider vector fields  $V \in \mathfrak{X}(M)$  which are critical points for the energy functional  $E|_{\mathfrak{X}^\rho(M)}$ , restricted to vector fields of the same constant length. The Euler-Lagrange equations of this variational condition are given by

$$(1.4) \quad \nabla^* \nabla V \text{ is collinear to } V.$$

As usual, for  $\rho \neq 0$  condition (1.4) is taken as a definition of critical points for the energy functional restricted to vector fields of the same length in the non-compact case. If  $V$  is a light-like vector field then (1.4) is still a sufficient condition so that  $V$  is a critical point for the energy functional  $E|_{\mathfrak{X}^0(M)}$ , restricted to light-like vector fields ([4], Theorem 26).

In [6], four-dimensional Einstein Lorentzian lie groups were classified into four types, denoted by 16 cases. In the present paper using a case-by-case argument we shall provide a complete investigation of harmonicity of vector fields on four-dimensional Einstein Lorentzian lie groups.

The paper is organized in the following way. In Section 2, we shall recall basic properties of Einstein Lorentzian Lie algebra, as described in [6]. Harmonicity properties of vector fields of four-dimensional Einstein Lorentzian Lie group of types (a1), (a2), (c1) and (c2) will be investigated in Sections 3-6, respectively. Finally, the energy and the minimality of all these vector fields is explicitly calculated in Section 7.

## 2. EINSTEIN LORENTZIAN LIE GROUPS

Let  $(G, g)$  be a four-dimensional Lorentzian Lie group. By Proposition 2.3 in [6], the Lie algebra  $\mathfrak{g}$  of  $G$  is a semi-direct product  $\mathfrak{t} \ltimes \mathfrak{g}_3$ , where  $\mathfrak{t} = \text{span}\{e_4\}$  acts on  $\mathfrak{g}_3 =$

$\text{span}\{e_1, e_2, e_3\}$ , and the Lorentzian inner product on  $\mathfrak{g}$  is described by

$$(a) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (b) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which conditions (a) and (b) here, are not exactly as same as the conditions (a) and (b) of Proposition 2.3 in [6]. Actually this corrects the misprint which has happened about conditions (a) and (b) in [6]. Let  $G$  be a four-dimensional simply connected Lie group. If  $g$  is a left-invariant Lorentzian Einstein metric on  $G$  and so, described by one of conditions (a), (b), (c), then by theorem 3.1 in [6] the Lie algebra  $\mathfrak{g}$  of  $G$  is isometric to  $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$ , where  $\mathfrak{g}_3 = \text{span}\{e_1, e_2, e_3\}$  and  $\mathfrak{r} = \text{span}\{e_4\}$ , and one of the following cases occurs.

(a)  $\{e_i\}_{i=1}^4$  is a pseudo-orthonormal basis, with  $e_3$  time-like. In this case,  $G$  is isometric to one of the following semi-direct products  $\mathbb{R} \ltimes G_3$ :

(a1)  $\mathbb{R} \ltimes H$ , where  $H$  is the Heisenberg group and  $\mathfrak{g}$  is described by one of the following sets of conditions :

- (1)  $[e_1, e_2] = \epsilon A e_1, [e_1, e_3] = A e_1, [e_1, e_4] = \delta A e_1, [e_3, e_4] = -2A\delta(\epsilon e_2 - e_3),$
- (2)  $[e_1, e_2] = \frac{\epsilon\sqrt{A^2-B^2}}{2}e_1, [e_1, e_3] = -\frac{\epsilon\delta\sqrt{A^2-B^2}}{2}e_1, [e_1, e_4] = \frac{\delta A+B}{2}e_1, [e_2, e_4] = B(e_2 + \delta e_3), [e_3, e_4] = A(e_2 + \delta e_3),$
- (3)  $[e_1, e_2] = \frac{\epsilon A\sqrt{A^2-B^2}}{B}e_1, [e_1, e_3] = \epsilon\sqrt{A^2-B^2}e_1, [e_2, e_4] = B e_2 - A e_3, [e_3, e_4] = A e_2 - \frac{A^2}{B}e_3,$
- (4)  $[e_1, e_2] = \epsilon\sqrt{A^2-B^2}e_1 + B e_2, [e_3, e_4] = A e_3,$

(a2)  $\mathbb{R} \ltimes \mathbb{R}^3$ , where  $\mathfrak{g}$  is described by one of the following sets of conditions:

- (5)  $[e_1, e_4] = -(A+B)e_1, [e_2, e_4] = B e_2 - \epsilon\sqrt{A^2+AB+B^2}e_3, [e_3, e_4] = \epsilon\sqrt{A^2+AB+B^2}e_2 + A e_3,$
- (6)  $[e_1, e_4] = -2A e_1, [e_2, e_4] = -5A e_2 + 6\epsilon A e_3, [e_3, e_4] = A e_3,$
- (7)  $[e_1, e_4] = A e_1, [e_2, e_4] = A e_2 + B e_3, [e_3, e_4] = B e_2 + A e_3,$
- (8)  $[e_1, e_4] = \epsilon\frac{A+B}{3}e_1, [e_2, e_4] = \epsilon\frac{5B-A}{6}e_2 + B e_3, [e_3, e_4] = A e_2 + \epsilon\frac{5A-B}{6}e_3,$
- (9)  $[e_1, e_4] = \frac{5A}{2}e_1 + 3\epsilon A e_3, [e_2, e_4] = A e_2, [e_3, e_4] = -\frac{A}{2}e_3,$
- (10)  $[e_1, e_4] = A e_1 + \epsilon\sqrt{B^2-A^2-C^2-AC}e_2, [e_2, e_4] = \epsilon\sqrt{B^2-A^2-C^2-AC}e_1 - (A+C)e_2 - B e_3, [e_3, e_4] = B e_2 + C e_3,$
- (11)  $[e_1, e_4] = -\frac{2\epsilon\sqrt{2}A}{3}e_1 + \delta A e_3, [e_2, e_4] = \frac{\epsilon\sqrt{2}A}{3}e_2, [e_3, e_4] = A e_2 - \frac{\epsilon\sqrt{2}A}{6}e_3,$

(c)  $\{e_i\}_{i=1}^4$  is a basis, with the inner product  $g$  on  $\mathfrak{g}$  completely determined by  $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_4) = g(e_4, e_3) = 1$  and  $g(e_i, e_j) = 0$  otherwise. In this case,  $G$  is isometric to one of the following semi-direct products  $\mathbb{R} \ltimes G_3$ :

(c1)  $\mathbb{R} \ltimes H$ , where  $\mathfrak{g}$  is described by one of the following sets of conditions

- (12)  $[e_1, e_2] = \epsilon(A+B)e_3, [e_1, e_4] = C e_1 + B e_2 + D e_3, [e_2, e_4] = B e_1 + E e_3, [e_3, e_4] = C e_3,$
- (13)  $[e_1, e_2] = B e_3, [e_1, e_4] = \frac{(C+D)^2-B^2}{4A}e_1 + D e_2 + F e_3, [e_2, e_4] = C e_1 + A e_2 + E e_3, [e_3, e_4] = \frac{(C+D)^2-B^2+4A^2}{4A}e_3,$
- (14)  $[e_1, e_2] = \epsilon\sqrt{((A+D)^2+4B^2)}e_3, [e_1, e_4] = -B e_1 + D e_2 + E e_3, [e_2, e_4] = A e_1 + B e_2 + C e_3,$

(c2)  $\mathbb{R} \ltimes \mathbb{R}^3$ , where  $\mathfrak{g}$  is described by one of the following sets of conditions:

$$(15) [e_1, e_2] = Ae_2 + Be_3, [e_2, e_4] = -Ae_1 + Ce_3,$$

$$(16) [e_1, e_4] = Ae_1 + Be_2 + Ce_3, [e_2, e_4] = De_1 + Ee_2 + Fe_3, [e_3, e_4] = \frac{(B+D)^2 + 2(A^2 + E^2)}{2(E+A)}e_3$$

In all the cases listed above,  $\epsilon = \pm 1$  and  $\delta = \pm 1$ . Each of Einstein examples for case (b) is isometric to one of cases listed in case (a).

### 3. HARMONICITY OF VECTOR FIELDS: TYPE (a1)

All four-dimensional simply connected, Einstein Lorentzian Lie groups of type (a1) are indeed symmetric [6] and the study of harmonic invariant vector fields on these spaces would be natural and interesting. Consider a four-dimensional simply connected Lie group  $G$  which is isometric to  $\mathbb{R} \ltimes H$ , a left-invariant Lorentzian Einstein metric  $g$  on  $G$  described by condition (a) and a pseudo-orthonormal basis  $\{e_i\}_{i=1}^4$ , with  $e_3$  time-like. With these hypotheses we have the following result.

**Theorem 3.1.** *Let  $\mathfrak{g}$  be described by one of the sets of conditions (1) – (4) and  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  be a left-invariant vector field on  $G$  for some real constants  $a, b, c, d$ . For the different conditions (1) – (4), (which  $V$  is restricted to vector fields of the same length), we have:*

- (1) :  $V$  is a critical point for the energy functional if and only if  $V = c(e_2 - e_3 - e_4)$ , that is,  $b = -c = -d$ . In this case  $\epsilon = 1$ ,  $\nabla^* \nabla V = 3A^2 V$ .
- (2) :  $V$  is a critical point for the energy functional if and only if  $V = c(e_2 + e_3 - e_4)$ , that is,  $b = c = -d$ . In this case  $\epsilon = -1$ ,  $\delta = 1$ ,  $\nabla^* \nabla V = -\frac{3}{4}(A + B)^2 V$ .
- (3) :  $V$  is a critical point for the energy functional, in this case,  $\nabla^* \nabla V = -\frac{(A^2 - B^2)^2}{B^2} V$ .
- (4) :  $V$  is a critical point for the energy functional if and only if  $V = ae_1 + be_2$ . In this case  $\nabla^* \nabla V = (B^2 - A^2)V$ .

*Proof.* The above statement is obtained from a case-by-case argument. As an example, we report the details for case (4) here. Let  $V$  be a critical point for the energy functional. The components of the Levi-Civita connection are the following:

$$(3.5) \quad \begin{aligned} \nabla_{e_1} e_1 &= -\epsilon \sqrt{A^2 - B^2} e_2, & \nabla_{e_1} e_2 &= \epsilon \sqrt{A^2 - B^2} e_1, \\ \nabla_{e_3} e_3 &= Ae_4, & \nabla_{e_3} e_4 &= Ae_3, \end{aligned}$$

while  $\nabla_{e_i} e_j = 0$  in the remaining cases.

By Equation (3.5) we obtain

$$(3.6) \quad \begin{aligned} \nabla_{e_1} V &= \epsilon \sqrt{A^2 - B^2} (be_1 - ae_2), & \nabla_{e_2} V &= 0, \\ \nabla_{e_3} V &= A(de_3 + ce_4), & \nabla_{e_4} V &= 0. \end{aligned}$$

From (3.6) it follows at once that there are no parallel vector fields  $V \neq 0$  belonging to  $\mathfrak{g}$ . We can now calculate  $\nabla_{e_i} \nabla_{e_i} V$  and  $\nabla_{\nabla_{e_i} e_i} V$  for all indices  $i$ . We obtain

$$(3.7) \quad \begin{aligned} \nabla_{e_1} \nabla_{e_1} V &= -(A^2 - B^2)(ae_1 + be_2), & \nabla_{e_2} \nabla_{e_2} V &= 0, \\ \nabla_{e_3} \nabla_{e_3} V &= A^2(ce_3 + de_4), & \nabla_{e_4} \nabla_{e_4} V &= 0, \\ \nabla_{\nabla_{e_1} e_1} V &= 0, & \nabla_{\nabla_{e_2} e_2} V &= 0, & \nabla_{\nabla_{e_3} e_3} V &= 0, & \nabla_{\nabla_{e_4} e_4} V &= 0. \end{aligned}$$

Thus, we find

$$\nabla^* \nabla V = \sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) = -(A^2 - B^2)(ae_1 + be_2) - A^2(ce_3 + de_4).$$

Since  $\nabla^*\nabla V = -(A^2 - B^2)V - B^2(ce_3 + de_4)$ , condition 1.4 results that  $c = d = 0$ . In the other direction, let  $V = (B^2 - A^2)(ae_1 + be_2)$ . A direct calculation yields that  $\nabla^*\nabla V = -(A^2 - B^2)V$ .  $\square$

Next, in this case suppose that  $V$  is a critical point for the energy functional vector field. Clearly,  $\nabla^*\nabla V = 0$  if and only if  $(A^2 - B^2)V = 0$  which means that  $A = \pm B$ . Let  $R$  denote the curvature tensor of  $(M, g)$ , taken with the sign convention  $R(X, Y) = \nabla[X, Y] - [\nabla X, \nabla Y]$ . Then, using (3.6), we find

$$\begin{aligned} R(\nabla_{e_1} V, V)e_1 &= \epsilon^3 \sqrt{(A^2 - B^2)^3} (a^2 + b^2)e_2, & R(\nabla_{e_2} V, V)e_2 &= 0, \\ R(\nabla_{e_3} V, V)e_3 &= 0, & R(\nabla_{e_4} V, V)e_4 &= 0. \end{aligned}$$

and so,

$$\text{tr}[R(\nabla \cdot V, V)] = \sum_i \epsilon_i R(\nabla_{e_i} V, V)e_i = \epsilon^3 \sqrt{(A^2 - B^2)^3} (a^2 + b^2)e_2.$$

Hence,  $\text{tr}[R(\nabla \cdot V, V)] = 0$  if and only if  $A = B$ . Applying this argument for other cases of type (a1) proves the following classification result.

**Theorem 3.2.** *Let  $G$  be a four-dimensional simply connected Lie group of type (a1) and  $V$  be a critical point for the energy functional restricted to vector fields of the same length, described by conditions (2) – (4) of theorem 3.1, then for different cases (2), (3) and (4),  $V$  defines harmonic map if and only if  $A = -B$ ,  $A = \pm B$  and  $A = \pm B$  respectively.*

A vector field  $V$  is geodesic if  $\nabla_V V = 0$ , and is Killing if  $\mathcal{L}_V g = 0$ , where  $\mathcal{L}$  denotes the Lie derivative. Parallel vector fields are both geodesic and Killing, and vector fields with these special geometric features often have particular harmonicity properties [1, 10, 12, 13]. A straightforward calculation proves the following main classification result.

**Corollary 3.3.** *Let  $G$  be a four-dimensional simply connected Lie group of type (a1),  $\mathfrak{g}$  be described by one of the sets of conditions (1) – (4) and  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$ . If  $g$  be a left-invariant Lorentzian Einstein metric on  $G$ , then for the different cases (1) – (4) the equivalent properties for  $V$  are listed in the following Table 1 (which  $V$  is restricted to vector fields of the same length).*

**Table 1:** The following properties for  $V$  on Lie group of type (a1) are equivalent.

$(G, g)$	Equivalent properties
(1)	$V$ is geodesic; $\equiv V$ is a critical point for the energy functional; $\equiv$ none of these vector fields is harmonic (in particular, defines a harmonic map); $\equiv V = c(e_2 - e_3 - e_4)$ ,
(2)	$V$ is geodesic; $\equiv V$ is harmonic if and only if $A = -B$ ; $\equiv V$ is a critical point for the energy functional; $\equiv V$ defines harmonic map if and only if $A = -B$ ; $\equiv V$ is Killing if and only if $A = -B$ and $d = 0$ ; $\equiv V = c(e_2 + e_3 - e_4)$ ,
(3)	$V$ is geodesic if and only if $A = \pm B$ and $b = \mp c$ ; $\equiv V$ is harmonic if and only if $A = \pm B$ ; $\equiv V$ is a critical point for the energy functional; $\equiv V$ defines harmonic map if and only if $A = \pm B$ ; $\equiv V$ is Killing if and only if $A = \pm B$ , $b = \mp c$ and $d = 0$ ,
(4)	$V$ is geodesic if and only if $A = \pm B$ ; $\equiv V$ is harmonic if and only if $A = \pm B$ ; $\equiv V$ is a critical point for the energy functional; $\equiv V$ defines harmonic map if and only if $A = \pm B$ ; $\equiv V$ is Killing if and only if $A = \pm B$ ; $\equiv V = ae_1 + be_2$ .

**Remark 3.4.** Recall that for a Lorentzian Lie group, a left-invariant vector field  $V$  is spatially harmonic if and only if

$$(3.8) \quad \tilde{X}_V = -\nabla^* \nabla V - \nabla_V \nabla_V V - \operatorname{div} V \cdot \nabla_V V + (\nabla V)^t \nabla_V V \text{ is collinear to } V.$$

Clearly, conditions (1.4) and (3.8) coincide for geodesic vector fields. Hence, the results listed in Table 1 show that for cases (1) and (2),  $V$  is spatially harmonic and for cases (3) and (4),  $V$  is spatially harmonic if and only if  $A = \pm B$ ,  $b = \mp c$  and  $A = \pm B$  respectively.

#### 4. HARMONICITY OF VECTOR FIELDS: TYPE (a2)

Let  $(G, g)$  be a four-dimensional generalized symmetric space of type (a2), where  $g$  is a left-invariant Lorentzian Einstein metric of condition (a). Suppose that  $\mathfrak{g}$  is described by one of the sets of conditions (5) – (11) and  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  is a left-invariant vector field on  $G$  for some real constants  $a, b, c, d$ . As an example for the sets of condition (7) a direct calculation yields that:

$$(4.9) \quad \begin{aligned} \nabla_{e_1} e_1 &= -\nabla_{e_1} e_4 = -A, & \nabla_{e_2} e_2 &= -\nabla_{e_2} e_4 = -A, \\ \nabla_{e_3} e_3 &= \nabla_{e_3} e_4 = A, & \nabla_{e_4} e_2 &= \nabla_{e_4} e_3 = -B. \end{aligned}$$

Using (4.9) to calculate  $\nabla_{e_i} V$  for all indices  $i$ , we get

$$(4.10) \quad \begin{aligned} \nabla_{e_1} V &= A(de_1 - ae_4), & \nabla_{e_2} V &= A(de_2 - be_4), \\ \nabla_{e_3} V &= A(de_3 + ce_4), & \nabla_{e_4} V &= -B(ce_2 + be_3). \end{aligned}$$

Clearly, there are no parallel vector fields  $V \neq 0$  in  $\mathfrak{g}$ . We can now use (4.10) to obtain  $\nabla_{e_i} \nabla_{e_i} V$  for all indices  $i$  and we find

$$(4.11) \quad \begin{aligned} \nabla_{e_1} \nabla_{e_1} V &= -A^2(ae_1 + de_4), & \nabla_{e_2} \nabla_{e_2} V &= -A^2(be_2 + de_4), \\ \nabla_{e_3} \nabla_{e_3} V &= A^2(ce_3 + de_4), & \nabla_{e_4} \nabla_{e_4} V &= B^2(be_2 + ce_3), \end{aligned}$$

and for  $\nabla_{\nabla_{e_i} e_i} V$  for all indices  $i$  we have

$$(4.12) \quad \nabla_{\nabla_{e_1} e_1} V = \nabla_{\nabla_{e_2} e_2} V = -\nabla_{\nabla_{e_3} e_3} V = AB(ce_2 + be_3), \quad \nabla_{\nabla_{e_4} e_4} V = 0,$$

Then, we get

$$(4.13) \quad \begin{aligned} \nabla^* \nabla V &= \sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) = \\ &= -A^2 ae_1 + ((B^2 - A^2)b - 3ABc)e_2 + ((B^2 - A^2)c - 3ABb)e_3 - 3A^2 de_4. \end{aligned}$$

Hence,  $\nabla^* \nabla V = (B^2 - A^2 - 3AB)V$  if and only if  $b = c$  and  $a = d = 0$ .

Using some similar argument for other cases lead to the following result.

**Theorem 4.1.** Let  $G$  be a four-dimensional simply connected Lie group of type (a2) and  $V = ae_1 + be_2 + ce_3 + de_4$  be a left-invariant vector field on  $G$  for some real constants  $a, b, c, d$ . For the different conditions (5) – (11), (which  $V$  is restricted to vector fields of the same length), we have:

- (5) :  $V$  is a critical point for the energy functional if and only if  $V = ae_1$ , that is,  $b = c = d = 0$ . In this case  $\nabla^* \nabla V = -(A + B)^2 V$ .
- (6) :  $V$  is a critical point for the energy functional if and only if  $V = b(e_2 - e_3)$ , that is,  $c = -b$ . In this case  $\epsilon = 1$ ,  $\nabla^* \nabla V = -13A^2 V$ .
- (7) :  $V$  is a critical point for the energy functional if and only if  $V = b(e_2 + e_3)$ , that is,  $c = b$ . In this case  $\nabla^* \nabla V = (B^2 - A^2 + 3AB)V$ .
- (8) :  $V$  is a critical point for the energy functional if and only if  $V = b(e_2 - e_3)$ , that is,  $c = -b$ . In this case  $\epsilon = -1$ ,  $\nabla^* \nabla V = \frac{13}{36}(A + B)^2 V$ .
- (9) :  $V$  is a critical point for the energy functional if and only if  $V = a(e_1 + e_3)$ , that is,  $c = a$ . In this case  $\epsilon = 1$ ,  $\nabla^* \nabla V = -\frac{13}{4}A^2 V$ .

- (10) :  $V$  is a critical point for the energy functional if and only if  $V = ce_3$ , that is,  $a = b = d = 0$ . In this case  $\nabla^* \nabla V = -C^2 V$ .
- (11) :  $V$  is a critical point for the energy functional if and only if  $V = c(\sqrt{2}e_1 - \frac{2}{3}\sqrt{2}e_2 + e_3)$ , that is,  $a = \sqrt{2}c$ ,  $b = -\frac{2}{3}\sqrt{2}c$ . In this case  $\epsilon = \delta = -1$ ,  $\nabla^* \nabla V = \frac{5}{18}A^2 V$ .

Again, for case (7), by (4.13), clearly  $\nabla^* \nabla V = 0$  if and only if  $a = d = 0$  and  $(B^2 - A^2 + 3AB) = 0$  which means that  $A = ((3 - \sqrt{13})/2)B$ . Using 4.10 with  $V = b(e_2 + e_3)$ , we find

$$\begin{aligned} R(\nabla_{e_2} V, V)e_2 &= A^2(A + 2B)c^2 e_4, \\ R(\nabla_{e_3} V, V)e_3 &= A^2(A + 2B)c^2 e_4, \\ R(\nabla_{e_1} V, V)e_1 &= R(\nabla_{e_4} V, V)e_4 = 0 \end{aligned}$$

and so,

$$\text{tr}[R(\nabla \cdot V, V)] = \sum_i \varepsilon_i R(\nabla_{e_i} V, V)e_i = 0.$$

Following this argument for other cases of type (a2) proves the following result.

**Theorem 4.2.** *Let  $G$  be a four-dimensional simply connected Lie group of type (a2) and  $V = ae_1 + be_2 + ce_3 + de_4$  be a critical point for the energy functional described by different conditions (5) – (11) of theorem 4.1, then it is easy to check that*

- for cases (5) and (8),  $V$  is a harmonic map if and only if  $A = -B$ .
- for case (7),  $V$  is a harmonic map if and only if  $A = ((-3 \pm \sqrt{13})/2)B$ .
- for case (10),  $V$  is a harmonic map if and only if  $C = B = 0$ .

Starting from (4.11), a straightforward calculation proves the following classification result.

**Corollary 4.3.** *Let  $V \in \mathfrak{g}$  be a left-invariant vector field on four-dimensional simply connected Lie group  $G$  of type (a2). If  $g$  be a left-invariant Lorentzian Einstein metric on  $G$ , then for the different cases (5) – (11) the equivalent properties for  $V$  are listed in the following Table 2 (which  $V$  is restricted to vector fields of the same length).*

By remark 3.4, the results listed in Table 2 show that for cases (6) – (9),  $V$  is spatially harmonic and for case (10),  $V$  is spatially harmonic if and only if  $C = 0$ .

## 5. HARMONICITY OF VECTOR FIELDS: TYPE (c1)

Consider a four-dimensional simply connected Lie group  $G$  of type (c1) and a basis  $\{X_i\}_{i=1}^4$ , with non-zero inner product  $g$  on  $\mathfrak{g}$  determined by  $g(X_1, X_1) = g(X_2, X_2) = g(X_3, X_4) = g(X_4, X_3) = 1$ . We can construct a pseudo-orthonormal frame field  $\{e_1, e_2, e_3, e_4\}$ , putting

$$(5.14) \quad e_1 = X_1, \quad e_2 = X_2, \quad e_3 = -(1/2)X_3 + X_4, \quad e_4 = (1/2)X_3 + X_4.$$

Clearly,  $e_3$  is time-like. A vector field  $V \in \mathfrak{g}$  is uniquely determined by its components with respect to the pseudo-orthonormal basis  $\{e_i\}$ . Hence  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  is a left-invariant vector field on  $G$  for some real constants  $a, b, c, d$ . Notice that the (constant) norm of  $V$  is given by  $\|V\|^2 = a^2 + b^2 - c^2 + d^2$ .

As a sample, for the sets of condition (14), using (5.14) we find

$$\begin{aligned} \nabla_{e_1} e_1 &= -B(e_3 - e_4), \quad \nabla_{e_1} e_2 = \frac{1}{2}(A + D - \epsilon\alpha)(e_3 - e_4), \\ \nabla_{e_1} e_3 &= \nabla_{e_1} e_4 = -Be_1 + \frac{1}{2}(A + D - \epsilon\alpha)e_2, \\ \nabla_{e_2} e_1 &= \frac{1}{2}(A + D + \epsilon\alpha)(e_3 - e_4), \quad \nabla_{e_2} e_2 = B(e_3 - e_4), \\ \nabla_{e_2} e_3 &= \nabla_{e_2} e_4 = \frac{1}{2}(A + D + \epsilon\alpha)e_1 + Be_2, \end{aligned} \quad (5.15)$$



**Table 2:** The following properties for  $V$  on Lie group of type (a2) are equivalent.

$(G, g)$	Equivalent properties
(5)	$V$ is harmonic if and only if $A = -B$ ; $\equiv$ $V$ is a critical point for the energy functional if and only if $V = ae_1$ ; $\equiv$ $V$ defines harmonic map if and only if $A = -B$ ; $\equiv$ $V$ is Killing if and only if $A = -B$ and $V = ae_1$ .
(6)	$V$ is geodesic; $\equiv$ $V$ is a critical point for the energy functional; $\equiv$ none of these vector fields is harmonic (in particular, defines a harmonic map); $\equiv$ $V = b(e_2 - e_3)$ .
(7)	$V$ is geodesic; $\equiv$ $V$ is harmonic if and only if $A = ((-3 \pm \sqrt{13})/2)B$ ; $\equiv$ $V$ is a critical point for the energy functional; $\equiv$ $V$ defines harmonic map if and only if $A = ((-3 \pm \sqrt{13})/2)B$ ; $\equiv$ $V$ is Killing if and only if $A = -B$ ; $\equiv$ $V = b(e_2 + e_3)$ .
(8)	$V$ is geodesic if and only if $A = -B$ and $c = -b$ ; $\equiv$ $V$ is harmonic if and only if $A = -B$ ; $\equiv$ $V$ is a critical point for the energy functional if and only if $c = -b, a = d = 0$ ; $\equiv$ $V$ defines harmonic map if and only if $A = -B$ ; $\equiv$ $V$ is Killing if and only if $A = -B$ and $d = 0$ ; $\equiv$ $V = ae_1 + be_2 + ce_3 + de_4$ .
(9)	$V$ is geodesic; $\equiv$ $V$ is a critical point for the energy functional; $\equiv$ none of these vector fields is harmonic (in particular, defines a harmonic map); $\equiv$ $V = a(e_1 + e_3)$ .
(10)	$V$ is geodesic if and only if $C = 0$ ; $\equiv$ $V$ is harmonic if and only if $C = 0$ ; $\equiv$ $V$ is a critical point for the energy functional; $\equiv$ $V$ defines harmonic map if and only if $C = 0$ ; $\equiv$ $V$ is Killing if and only if $C = 0$ ; $\equiv$ $V = ce_3$ .
(11)	$V$ is a critical point for the energy functional; $\equiv$ none of these vector fields is harmonic (in particular, defines a harmonic map); $\equiv$ $V = c(\sqrt{2}e_1 - \frac{2}{3}\sqrt{2}e_2 + e_3)$ .

$$\begin{aligned}
\nabla_{e_4} e_1 &= \nabla_{e_3} e_1 = \frac{1}{2}(A - D - \epsilon\alpha)e_2 + E(e_3 - e_4), \\
\nabla_{e_4} e_2 &= \nabla_{e_3} e_2 = \frac{1}{2}(-A + D - \epsilon\alpha)e_1 + C(e_3 - e_4), \\
\nabla_{e_4} e_3 &= \nabla_{e_4} e_4 = \nabla_{e_3} e_3 = \nabla_{e_3} e_4 = Ee_1 + Ce_2,
\end{aligned}$$

where  $\alpha = \sqrt{(A + D)^2 + 4B^2}$ . Set  $u = e_3 - e_4$ . Then, from (5.15) we get  $\nabla_{e_i} u = 0$  for all indices  $i$ . Therefore,  $u$  is a parallel light-like vector field. The existence of a light-like parallel vector field is an interesting phenomenon which has no Riemannian counterpart, and characterizes a class of pseudo-Riemannian manifolds which illustrate many of differences between Riemannian and pseudo-Riemannian settings (see for example [7],[8]). For an arbitrary left-invariant vector field  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  we can now use (5.15) to calculate  $\nabla_{e_i} V$  for all indices  $i$ . We get

$$\begin{aligned}
\nabla_{e_1} V &= (c + d)(-Be_1 + \frac{1}{2}(D + A - \beta)e_2) \\
&\quad + (\frac{1}{2}(A + D - \beta)b - Ba)u, \\
\nabla_{e_2} V &= (c + d)(\frac{1}{2}(D + A + \beta)e_1 + Be_2) \\
&\quad + (\frac{1}{2}(A + D + \beta)a - Bb)u, \\
\nabla_{e_3} V &= \nabla_{e_4} V = (E(c + d) + \frac{1}{2}(D - A + \beta)b)e_1 \\
&\quad + (C(c + d) - (\frac{1}{2}(D - A + \beta)a)e_2 + (Cb + Ea)u,
\end{aligned} \tag{5.16}$$

where  $\beta = \epsilon\sqrt{(A + D)^2 + 4B^2}$ . Using (5.16), we calculate  $\nabla_{e_i} \nabla_{e_i} V$  and  $\nabla_{\nabla_{e_i} e_i} V$  and we find

$$\begin{aligned}
\nabla_{e_1} \nabla_{e_1} V &= -B(c + d)e_1 + \frac{1}{2}(A + T - \beta)(c + d)e_2 + (b\frac{1}{2}(A + T - \beta) - aB)u, \\
\nabla_{e_2} \nabla_{e_2} V &= \frac{1}{2}(A + T + \beta)(c + d)e_1 + B(c + d)e_2 + (a\frac{1}{2}(A + T + \beta) + bB)u,
\end{aligned} \tag{5.17}$$



$$\begin{aligned}
& \nabla_{e_3} \nabla_{e_3} V = \nabla_{e_4} \nabla_{e_4} V = \\
& -\frac{1}{4}(A - T - \beta)^2(e_1 + e_2) + \frac{1}{2}(Ca - Eb)(A - T - \beta)^2 u, \\
& \nabla_{\nabla_{e_1} e_1} V = \nabla_{\nabla_{e_2} e_2} V = 0, \\
(5.18) \quad & \nabla_{\nabla_{e_3} e_3} V = \nabla_{\nabla_{e_4} e_4} V = (\frac{1}{2}C(A + T + \beta) - BE)(c + d)e_1 + \\
& (BC + \frac{1}{2}E(A + T - \beta)(c + d)e_4 + (b(BC + \frac{1}{2}E(A + T - \beta)) + \\
& a(\frac{1}{2}C(A + T + \beta) - BE))u,
\end{aligned}$$

where  $\beta = \epsilon \sqrt{(A + D)^2 + 4B^2}$ . Hence from (5.17) and (5.18) we conclude that

$$\nabla^* \nabla V = \sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) = ((A + D)^2 + 4B^2)(c + d)u.$$

that is,  $\nabla^* \nabla V$  identically vanishes if and only if  $c = -d$ . Using (5.16), the curvature tensor is completely determined by

$$\begin{aligned}
R(\nabla_{e_1} V, V)e_1 &= -R(\nabla_{e_2} V, V)e_2 = \frac{1}{2}B(c + d)^2((A + D)^2 + 4B^2 + (D - A)\epsilon\alpha)u, \\
R(\nabla_{e_3} V, V)e_3 &= R(\nabla_{e_4} V, V)e_4 = \frac{1}{4}B(c + d)((A + D)^2 + 4B^2 + (D - A)\epsilon\alpha) \\
& (-2E(c + d) - b(D - A + \epsilon\alpha))e_1 - (2C(c + d) - a(D - A + \epsilon\alpha))e_2,
\end{aligned}$$

where  $\alpha = \sqrt{(A + D)^2 + 4B^2}$ . Therefore

$$\text{tr}[R(\nabla \cdot V, V)] = \sum_i \varepsilon_i R(\nabla_{e_i} V, V)e_i = 0.$$

Applying some similar argument for other cases lead to the following result.

**Theorem 5.1.** *All vector fields  $V \in \mathfrak{g}$  of a four-dimensional simply connected Lie group of type (c1), for the different conditions (12), (13) and (14), are critical points for the energy functional restricted to vector fields of the same length. Moreover, a vector field  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  defines a harmonic map if and only if  $V = ae_1 + be_2 + cu$ , that is,  $d = -c$ .*

Therefore, for type (c1) in all cases, left-invariant harmonic vector fields define harmonic maps, form three-parameter families.

Also, with regard to harmonicity properties of invariant vector fields, four-dimensional simply connected Lie groups of type (c1) display some particular features. The main geometrical reasons for the special behaviour of these groups are the existence of a parallel light-like vector field. Starting from (5.15), we can easily prove the following classification result.

**Proposition 5.2.** *Among four-dimensional Ricci-parallel Lorentzian Lie groups, type (c1) up to isometries, Let  $G$  be a four-dimensional simply connected Lie group of and  $V \in \mathfrak{g}$  be a left-invariant vector field on  $G$ . If  $g$  be a left-invariant Lorentzian Einstein metric on  $G$ , then we have the following classification.*

**Table 3 :** Geodesic, killing and parallel vector fields on  $G$ : type (c1).

$(G, g)$	Geodesic vector fields	Killing vector fields	parallel vector fields
(12)	$V = be_2 + cu$	$V = cu \Leftrightarrow C = 0$	$V = cu \Leftrightarrow C = 0$
(13)	$V = ae_1 - \frac{B-C-T}{2A}ae_2 + cu$	$V = cu \Leftrightarrow 4A^2 = B^2 - (C + T)^2$	$V = cu \Leftrightarrow 4A^2 = B^2 - (C + T)^2$
(14)	$V = cu$	$V = cu$	$V = cu$

Comparing Proposition 5.2 with Theorem 5.1, one sees the following main result which emphasizes once again the special role played by the parallel vector field  $u$ .

**Corollary 5.3.** *Let  $V \in \mathfrak{g}$  be a left-invariant vector field on four-dimensional simply connected Lie group  $G$  of type (c1). If  $g$  be a left-invariant Lorentzian Einstein metric on  $G$ , then for the*

different cases (12) – (14) the equivalent properties for  $V$  are listed in the following Table 4 (which  $V$  is restricted to vector fields of the same length).

**Table 4 :** The following properties for  $V$  Lie group of type (c1) are equivalent.

$(G, g)$	Equivalent properties
(12)	$V$ is geodesic if and only if $a = 0$ ; $\equiv V$ is harmonic; $\equiv V$ is a critical point for the energy functional $\equiv V$ defines harmonic map; $\equiv V$ is Killing if and only if $a = b = C = 0$ , that is, $V$ is collinear to $u$ ; $\equiv V$ is parallel if and only if $a = b = C = 0$ , that is, $V$ is collinear to $u$ ; $\equiv V = ae_1 + be_2 + cu$ ,
(13)	$V$ is geodesic if and only if $b = -\frac{B-C-T}{2A}a$ ; $\equiv V$ is harmonic; $\equiv V$ is a critical point for the energy functional $\equiv V$ defines harmonic map; $\equiv V$ is Killing if and only if $a = b = 0$ , $4A^2 = B^2 - (C + T)^2$ , that is, $V$ is collinear to $u$ ; $\equiv V$ is parallel if and only if $a = b = 0$ , $4A^2 = B^2 - (C + T)^2$ ; $\equiv V = ae_1 + be_2 + cu$ ,
(14)	$V$ is geodesic if and only if $a = b = 0$ ; $\equiv V$ is harmonic; $\equiv V$ is a critical point for the energy functional $\equiv V$ defines harmonic map; $\equiv V$ is Killing if and only if $a = b = 0$ , $\equiv V$ is parallel if and only if $a = b = 0$ ; $\equiv V = ae_1 + be_2 + cu$ ,

From theorem 5.3 it is easily seen that for cases (12), (13) and (14),  $V$  is spatially harmonic if and only if  $a = 0$ ,  $b = -\frac{B-C-T}{2A}a$  and  $a = b = 0$  respectively.

## 6. HARMONICITY OF VECTOR FIELDS: TYPE (c2)

We start classifying left-invariant vector fields on four-dimensional simply connected Lie group  $G$  of type (c2), proving the following result.

**Theorem 6.1.** *All vector fields  $V \in \mathfrak{g}$  of a four-dimensional simply connected Lie group of type (c2) are a critical point for the energy functional restricted to vector fields of the same length. Moreover, All vector fields  $V \in \mathfrak{g}$  of a four-dimensional simply connected Lie group of type (c2), case (15) defines a harmonic map and for case (16), vector field  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  defines a harmonic map if and only if  $c = -d$ .*

*Proof.* For the sets of condition (15), using (5.14) we deduce

$$\begin{aligned}
 \nabla_{e_3}e_1 &= \nabla_{e_4}e_1 = -Ae_2 + Be_3 - Be_4, \\
 \nabla_{e_3}e_2 &= \nabla_{e_4}e_2 = Ae_1 + Ce_3 - Ce_4, \\
 \nabla_{e_3}e_3 &= \nabla_{e_4}e_3 = \nabla_{e_3}e_4 = \nabla_{e_4}e_4 = Be_1 + Ce_2,
 \end{aligned}
 \tag{6.19}$$

while  $\nabla_{e_i}e_j = 0$  in the remaining cases. Let  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  be a vector field. From (6.19) we then get

$$\begin{aligned}
 \nabla_{e_1}V &= \nabla_{e_2}V = 0, \\
 \nabla_{e_3}V &= \nabla_{e_4}V = (Ab + Bc + Bd)e_1 \\
 &\quad + (-Aa + Cc + Cd)e_2 + (Ba + Cb)e_3 - (Ba + Cb)e_4 \\
 &= (Ab + Bc + Bd)e_1 + (-Aa + Cc + Cd)e_2 + (Ba + Cb)u.
 \end{aligned}
 \tag{6.20}$$

Thus,  $u$  is a parallel vector field, where we note the special role of  $u = e_3 - e_4$ . We calculate  $\nabla_{e_i}\nabla_{e_i}V$  and  $\nabla_{\nabla_{e_i}e_i}V$  and we find

$$\begin{aligned}
 \nabla_{e_1}\nabla_{e_1}V &= \nabla_{e_2}\nabla_{e_2}V = 0, \\
 \nabla_{e_3}\nabla_{e_3}V &= \nabla_{e_4}\nabla_{e_4}V = A(-Aa + Cc + Cd)e_1 \\
 &\quad - A(Ab + Bc + Bd)e_2 + (C(-Aa + Cc + Cd) + B(Ab + Bc + Bd))u, \\
 \nabla_{\nabla_{e_1}e_1}V &= \nabla_{\nabla_{e_2}e_2}V = \nabla_{\nabla_{e_3}e_3}V = \nabla_{\nabla_{e_4}e_4}V = 0,
 \end{aligned}
 \tag{6.21}$$

Hence, from (6.21) we conclude that

$$\nabla^* \nabla V = \sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) = 0.$$

Using (6.20), the curvature tensor is completely determined by

$$R(\nabla_{e_1} V, V)e_1 = R(\nabla_{e_2} V, V)e_2 = R(\nabla_{e_3} V, V)e_3 = R(\nabla_{e_4} V, V)e_4 = 0,$$

Therefore

$$\text{tr}[R(\nabla \cdot V, V)] = \sum_i \varepsilon_i R(\nabla_{e_i} V, V)e_i = 0.$$

A similar argument for case (16) will prove the statement.  $\square$

Using (6.21), with regard to geodesic and Killing vector fields we obtain the following.

**Proposition 6.2.** *Let  $G$  be a four-dimensional simply connected Lie group of type (c2) and  $V \in \mathfrak{g}$  be a left-invariant vector field on  $G$ . If  $g$  be a left-invariant Lorentzian Einstein metric on  $G$ , then we have the following Table 5.*

**Table 5 :** Geodesic, killing and parallel vector fields on  $G$ : type (c2).

$(G, g)$	Geodesic vector fields	Killing vector fields	parallel vector fields
(15)	$V = ae_1 + be_2 + cu$	$V = cu$	$V = cu$
(16)	$V = a(e_1 - e_2) + cu$	$V = a(e_1 - e_2)$	$\times$

Using Proposition 6.2 with Theorem 6.1, a straight forward calculation proves the following classification result.

**Corollary 6.3.** *Let  $V \in \mathfrak{g}$  be a left-invariant vector field on four-dimensional simply connected Lie group  $G$  of type (c2). If  $g$  be a left-invariant Lorentzian Einstein metric on  $G$ , then for the different cases (15) and (16) the equivalent properties for  $V$  are listed in the following Table 6 (which  $V$  is restricted to vector fields of the same length):*

**Table 6 :** The following properties for  $V$  on Lie group of type (c2) are equivalent.

$(G, g)$	Equivalent properties
(15)	$V$ is geodesic if and only if $d = -c$ ; $\equiv V$ is harmonic; $\equiv V$ is a critical point for the energy functional $\equiv V$ defines harmonic map; $\equiv V$ is Killing if and only if $a = b = 0, d = -c$ ; $\equiv V$ is parallel if and only if $a = b = 0, d = -c$ ; $\equiv V = ae_1 + be_2 + ce_3 + de_4$ ,
(16)	$V$ is geodesic if and only if $b = -a$ ; $\equiv V$ is harmonic; $\equiv V$ is a critical point for the energy functional $\equiv V$ defines harmonic map; $\equiv V$ is Killing if and only if $b = -a$ and $c = d = 0$ ; $\equiv V = ae_1 + be_2 + cu$ ,

Clearly by remark 3.4, the results listed in Table 6 show that for cases (15) and (16),  $V$  is spatially harmonic if and only if  $d = -c$ .

## 7. THE ENERGY OF VECTOR FIELDS

We calculate explicitly the energy of a vector field  $V \in \mathfrak{g}$  of a four-dimensional Einstein Lorentzian Lie group. This gives us the opportunity to determine some critical values of the energy functional on four-dimensional Einstein Lorentzian Lie group. We shall first discuss geometric properties of the map  $V$  defined by a vector field  $V \in \mathfrak{g}$ .

**7.1. Types (a1) and (a2).** Let  $(G, g)$  be a four-dimensional Einstein Lorentzian Lie group of types (a1) or (a2) and  $\{e_i\}_{i=1}^4$  be a pseudo-orthonormal basis with  $e_3$  time-like. We now prove the following.

**Proposition 7.1.** *Let  $G$  be a four-dimensional simply connected Lie group of type (a1) or (a2),  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  be a vector field on  $G$  and  $\mathcal{D}$  be its relatively compact domain. Denote by  $E_{\mathcal{D}}(V)$  the energy of  $V|_{\mathcal{D}}$ . For the different cases (1) – (11) the Energy of vector field  $V$  is listed in the following Table 7.*

**Table 7:** Energy of vector fields: types (a1) and (a2).

$(G, g)$	$E_{\mathcal{D}}(V)$
(1)	$(2 + A^2((\ V\ ^2 + 2(d^2 - b^2) + 2\delta d(b + c) - 2bc)/2)\text{vol}\mathcal{D})$
(2)	$(2 + (A + B)^2((a^2 + 3d^2)(A + B) + (B - A)(b - c)^2 - 2d(b - c)\sqrt{A^2 - B^2})/8)\text{vol}\mathcal{D}$
(3)	$(2 + \frac{(A-B)^2(A+B)^2}{2B^2}\ V\ ^2)\text{vol}\mathcal{D}$
(4)	$(2 + (A^2\ V\ ^2 - B^2(a^2 + b^2))/2)\text{vol}\mathcal{D}$
(5)	$(2 + (A + B)(A(a^2 - b^2) + 2\epsilon bc\sqrt{A^2 + AB + B^2 + B(a^2 + c^2)})/2)\text{vol}\mathcal{D}$
(6)	$(2 + (A^2(4a^2 + 12d^2 + 17c^2 + 24bc + 7b^2)/2))\text{vol}\mathcal{D}$ , in this case $\epsilon = 1$
(7)	$(2 + (A^2(\ V\ ^2 + 2d) - B^2(b^2 - c^2))/2)\text{vol}\mathcal{D}$
(8)	$(2 + (1/72)(A + B)(A(4a^2 - 17b^2 - 7c^2 + 12d^2 - 24bc) + B(4a^2 + 7b^2 + 17c^2 + 12d^2 + 24bc)))\text{vol}\mathcal{D}$ , in this case $\epsilon = -1$
(9)	$(2 + (1/2)A^2(\frac{7}{4}a^2 + b^2 + \frac{17}{4}c^2 + 3d^2 - 6ac))\text{vol}\mathcal{D}$ , in this case $\epsilon = 1$
(10)	$(2 - (1/2)(AC(a^2 - b^2) - C^2(a^2 + c^2) - 2\epsilon ab\sqrt{-A^2 - C^2 - ACC}))\text{vol}\mathcal{D}$ , in this case $B = 0$
(11)	$(2 + (1/36)A^2(3a^2 - b^2 + 17c^2 - 5d^2 - 10ab + 9\sqrt{2}ac - 9\sqrt{2}bc))\text{vol}\mathcal{D}$ , in this case $\epsilon = \delta = -1$

*Proof.* Let  $(G, g)$  be a pseudo-Riemannian manifold of dimension 4. Consider a local pseudo-orthonormal basis  $\{e_1, \dots, e_n\}$  of vector fields, with  $\varepsilon_i = g(e_i, e_i) = \pm 1$  for all indices  $i$ . Then, locally,

$$\|\nabla V\|^2 = \sum_{i=1}^n \varepsilon_i g(\nabla_{e_i} V, \nabla_{e_i} V).$$

These conclusions are obtained from a case-by-case argument. As an example, if  $V \in \mathfrak{g}$  is a vector field of a four-dimensional Einstein Lorentzian Lie group of type (a1), case (4), then (3.6) easily yields

$$\|\nabla V\|^2 = A^2\|V\|^2 - B^2(a^2 + b^2).$$

Therefore,  $\|\nabla V\| = 0$  if and only if  $A = B$ . Thus, if  $A = B$  vector fields of the same length, will minimize the energy.  $\square$

We already know from Theorems 3.1 and 4.1 which vector fields in  $\mathfrak{g}$  of Einstein Lorentzian Lie group are critical points for the energy functional. Taking into account Propositions (7.1), we then have the following.

**Theorem 7.2.** *Let  $(G, g)$  be a four-dimensional Einstein Lorentzian Lie group of (a1) or (a2) and  $\mathcal{D}$  a relatively compact domain of  $G$ , then one of the following cases occurs.*

- (1) :  $(2 - \frac{3}{2}A^2c^2)\text{vol}\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = c(e_2 - e_3 - e_4)$ . In this case  $\epsilon = 1$ .

- (2) :  $(2 + \frac{3}{8}(A+B)^2c^2)vol\mathcal{D}$  is the minimum value of the energy functional  $E_{\mathcal{D}}$ . This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = c(e_2 + e_3 - e_4)$ . In this case  $\epsilon = -\delta = -1$ .
- (3) :  $(2 - \frac{(A^2-B^2)^2}{2B^2}\rho^2)vol\mathcal{D}$  is the minimum value of the energy functional  $E_{\mathcal{D}}$  restricted to vector fields of constant length  $\rho$ . Such a minimum is attained by all vector fields  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  of length  $\|V\| = \rho = \sqrt{a^2 + b^2 - c^2 + d^2}$ .
- (4) :  $(2 + \frac{(A^2-B^2)}{2}\rho^2)vol\mathcal{D}$  is the minimum value of the energy functional  $E_{\mathcal{D}}$  restricted to vector fields of constant length  $\rho$ . Such a minimum is attained by all vector fields  $V = ae_1 + be_2 \in \mathfrak{g}$  of length  $\|V\| = \rho = \sqrt{a^2 + b^2}$ .
- (5) :  $(2 + \frac{(A+B)^2}{2}\rho^2)vol\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = ae_1$ .
- (6) :  $2vol\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = b(e_2 - e_3)$ . In this case  $\epsilon = 1$ .
- (7) :  $2vol\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = b(e_2 + e_3)$ . Clearly in this case  $\|V\|^2 = b^2 - b^2 = 0$ .
- (8) :  $2vol\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = b(e_2 - e_3)$ . In this case  $\epsilon = -1$ .
- (9) :  $2vol\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = b(e_1 + e_3)$ . In this case  $\epsilon = 1$ .
- (10) :  $(2 - \frac{C^2}{2}\rho^2)vol\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = ce_3$ . In this case  $B = 0$ .
- (11) :  $(2 - \frac{589}{324}A^2c^2)vol\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = c(\sqrt{2}e_1 - \frac{2}{3}\sqrt{2}e_2 + e_3)$ . In this case  $\epsilon = \delta = -1$ .

*Proof.* The above statement is obtained from a case-by-case argument. By proposition 7.1, for case (4)

$$(7.22) \quad E_{\mathcal{D}}(V) = (2 + (A^2\|V\|^2 - B^2(a^2 + b^2))/2)vol\mathcal{D}.$$

On the other hand, from theorem 3.1,  $V = ae_1 + be_2$ . Substituting  $V$  in (7.22), we find  $E_{\mathcal{D}}(V) = (2 + \frac{(A^2-B^2)}{2}(a^2 + b^2))vol\mathcal{D}$ , which ends the proof.  $\square$

**7.2. Types (c1) and (c2).** Let  $(G, g)$  be a four-dimensional Einstein Lorentzian Lie group of type (c1) or (c2) and  $\{e_1, \dots, e_4\}$  be a local pseudo-orthonormal basis of vector fields described in 5.14. We verify the following.

**Proposition 7.3.** *Let  $G$  be a four-dimensional simply connected Lie group of type (c1) or (c2),  $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$  be a vector field on  $G$  and  $\mathcal{D}$  be its relatively compact domain. Denote by  $E_{\mathcal{D}}(V)$  the energy of  $V|_{\mathcal{D}}$ . For the different cases (12) – (16) the Energy of vector field  $V$  is listed in the following Table 8.*

**Table 8:** Energy of vector fields: types (c1) and (c2).

$(G, g)$	$E_{\mathcal{D}}(V)$
(12)	$(2 + ((A + B)^2 + C^2)(c + d)^2/2))vol\mathcal{D}$
(13)	$(2 + (B^2 + 4A^2 - 2CB - 2BD + (C + D)^2)(B^2 + 4A^2 + 2CB + 2BD + (C + D)^2)(c + d)^2/32A^2))vol\mathcal{D}$
(14)	$(2 + ((A + D)^2 + 4B^2)(c + d)^2/2))vol\mathcal{D}$
(15)	$2vol\mathcal{D}$
(16)	$(2 + ((A + B)^2 + A^2 + B^2)(c + d)^2/2))vol\mathcal{D}$

*Proof.* We followed the same argument used in Proposition 7.1 for case (14). From equation (6.20), we deduce

$$||\nabla V||^2 = ((A + D)^2 + 4B^2)(c + d)^2.$$

Therefore,  $||\nabla V|| = 0$  if and only if  $c = -d$ . Therefore, if  $c = -d$  vector fields of the same length, will minimize the energy.  $\square$

Theorems 5.1 and 6.1 show us which vector fields in  $\mathfrak{g}$  of Einstein Lorentzian Lie group of types (c1) and (c2), are critical points for the energy functional. Thus by Proposition (7.3), we then have the following result.

**Theorem 7.4.** *Let  $(G, g)$  be a four-dimensional Einstein Lorentzian Lie group of types (c1) and (c2) and  $\mathcal{D}$  a relatively compact domain of  $G$ , then one of the following cases occurs.*

- (12), (13), (14) and (16):  $2vol\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form  $V = ae_1 + be_2 + cu$ , where  $\{e_i\}$  is the base described in (5.14).
- (15) :  $2vol\mathcal{D}$  is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by every invariant vector fields  $V$ .

**Acknowledgements.** The author wishes to express his sincere gratitude toward the professor Giovanni Calvaruso for his valuable remarks and comments.

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